DE-Vol. 46, Mechanical Design and Synthesis
ASME 1992

A CONSTRUCTIVE VARIATIONAL GEOMETRY BASED MECHANISM DESIGN SOFTWARE PACKAGE

Philip H. Todd
Saltire Software
Beaverton, Oregon

ABSTRACT

This paper describes the design and implementation of a variational geometry based mechanism design and analysis software package. A new, constructive approach to variational geometry allows positions and their derivatives to be computed using closed form expressions rather than an iterative procedure. This facilitates an interactive kinematics model. Statics and dynamics capabilities are added by means of the principle of least work and d'Alambert's principle.

Examples are presented which highlight the flexibility and ease of use of the system.

1. INTRODUCTION

Traditional dynamics software packages (Chace 1984), while powerful in their analysis capabilities are somewhat lacking in their ease of use. They are most suitable for use when a design is complete to perform "soft prototyping" of a mechanism. In practice they are typically used by expert analysts who have been extensively trained on the software. Mechanism synthesis packages (Mittelstadt et al., 1985), on the other hand, while easy to use, are limited in the types of problem which they are designed to handle.

A third type of package which has recently emerged as a tool for the early or "conceptual" phase of mechanical design is the variational geometry system (Lin et al., 1981, Light and Gossard, 1982). In this paper, we describe a variational geometry approach to mechanism design and analysis. The technical basis for this approach is a new method for solving constraint systems, called "Constructive Variational Geometry". This method lends itself to incorporating engineering mechanics into a variational geometry system and provides an extremely flexible and easy to use tool for the conceptual phase of mechanism design.

2. CONSTRUCTIVE VARIATIONAL GEOMETRY

A variational geometry system can be viewed as providing a mapping from a set of constraint values **u** to a set of geometric locations **x(u)**. Conventionally, this mapping may involve numerical analysis. The author has developed a technique called Constructive Variational Geometry (CVG) (Todd, 1987, 1989) which uses graph

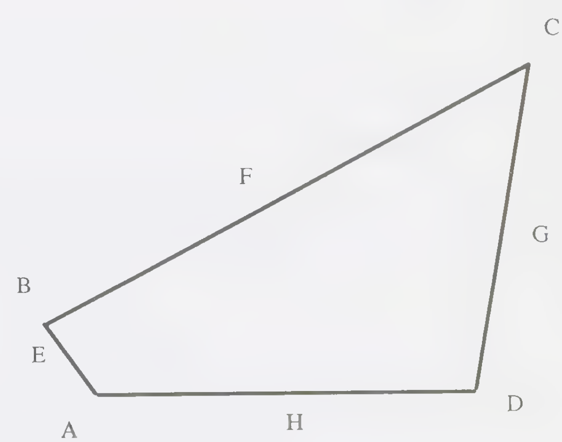

FIGURE 1: This four bar linkage is specified by the angle DAB=60 degrees and by the following lengths: AB=2; BC=CD=DA=3

theory and elementary geometric constructions to derive a purely analytic technique for generating this map.

Graph theoretical techniques described in Todd (1989) are used to convert a dimensioned sketch into a recipe for constructing the geometry. This recipe comprises a sequence of elementary constructions, each of which is responsible for constructing a single piece of geometry from its specified relation to known (already constructed) geometry.

As an example, we describe a recipe to construct the four bar linkage drawn in figure 1.

Construct a point (point A) at the origin.

Construct a line (line H) through point A in the direction of the x-axis.

Construct a line (line E) through point A angle 60 degrees to line H.

Construct a point (point B) on line E distance 1 from point A.

Construct a point (point D) on line F distance 3 from point A.

Construct a point (point C) distance 3 from point B and distance 3 from point D.

Construct a line (line F) through points B and C.

Construct a line (line G) through points C and D.

Each individual construction which is part of the recipe described above has a corresponding set of equations which may be solved to determine the unknown geometrical entity.

For example, points B and D are determined by the construction which finds a point (x,y) lying on the line ax+by+c=0 and distance r from point (x_0, y_0).

An equation for (x,y) can be found by substituting the linear equation :

$$ax+by+c = 0 \qquad (1)$$

into the quadratic

$$(x-x_0)^2 + (y-y_0)^2 = r^2 \qquad (2)$$

A more geometric formulation of these equations, however allows us to refer to the sketch in order to select the appropriate solution. We create the point (x_1, y_1) which is the projection of point (x_0, y_0) on the line ax+by+c=0.

$$(x_1, y_1) = (x_0, y_0) - (ax_0+by_0+c)\ (a,b) \qquad (3)$$

The distance between (x_1, y_1) and (x_0, y_0) is the same as the perpendicular distance between (x_0, y_0) and the line ax+by+c=0, which is $|ax_0+by_0+c|$. Hence the point (x,y) is distance $d = \mathrm{sqrt}(r^2 - (ax_0+by_0+c)^2)$ along the line ax+by+c = 0 from (x_1, y_1). Hence

$$(x,y) = (x_1, y_1) + d(b,-a) \qquad (4)$$

$$\text{or } (x_1, y_1) - d(b,-a) \qquad (5)$$

The choice of which solution to use is made by determining which solution reflects the ordering of the points and lines present in the user's sketch.

Similar algebraic solutions exist to the other constructions. Each of these constructions can be expressed algebraically as a set of linear equations along with, at worst, a single quadratic (Todd and Cherry, 1989). As such all are soluble using analytic techniques. Applying the appropriate constructions in the sequence determined by the aforementioned graph-analysis, allows the Constructive Variational Geometry system to solve for the geometry as a function of the constraint values.

Let **u** be the vector of constraint values. Let there be m points of interest in our drawing and n lines of interest (a point of interest is a mass center or a point of force application, a line of interest is a line with non zero moment of inertia or a line of torque application). The above geometric analysis yields the position of point i $(1 \le i \le m)$ as a function of these constraints: $(x_i(\mathbf{u}), y_i(\mathbf{u}))$. The angle of line j $(1 \le j \le n)$ may also be derived again as a function of the free constraints: $\theta_j(\mathbf{u})$.

Let **x** be the generalized position vector defined as follows:

$$\begin{aligned} \mathbf{x}_{2i-1}(\mathbf{u}) &= x_i(\mathbf{u}) && (1 \le i \le m) && (6)\\ \mathbf{x}_{2i}(\mathbf{u}) &= y_i(\mathbf{u}) && (1 \le i \le m) \\ \mathbf{x}_{2m+j}(\mathbf{u}) &= \theta_j(\mathbf{u}) && (1 \le j \le n) \end{aligned}$$

The CVG system provides the function:

$$\mathbf{x}(\mathbf{u}) \qquad (7)$$

By differentiating the sets of equations representing each individual construction, one can derive a second set of equations describing the velocity of the unknown point or line. As the equations for our geometrical constructions are, at worst, quadratic, differentiating them yields linear equations for the unknown velocity.

For example differentiating the equations given above for the construction of a point (x,y) incident to the line ax+by+c=0 and distance r from point (x_0, y_0) yields:

$$ax' + by' + a'x + b'y + c' = 0 \qquad (8)$$

$$2(x-x_0)x' + 2(y-y_0)y' + 2(x-x_0)x_0' + 2(y-y_0)y_0' = 2rr' \qquad (9)$$

For given values of (x,y) , (x_0, y_0), (a,b,c), and r these are linear equations for (x',y') in terms of (x_0', y_0'), (a',b',c') and r'.

By concatenating such velocity constructions according to the same sequence as the geometry, the CVG system is able to evaluate the function:

$$\mathbf{x}'(\mathbf{u},\mathbf{u}') \tag{10}$$

Repeating the above step for second derivatives, allows the system to evaluate the function

$$\mathbf{x}''(\mathbf{u},\mathbf{u}',\mathbf{u}'') \tag{11}$$

For given values of (x,y) , (x_0,y_0), (a,b,c), (x',y'), (x_0',y_0'), (a',b',c'), r and r' these are linear equations for (x'',y'') in terms of (x_0'',y_0''), (a'',b'',c'') and r''.

Substituting $\mathbf{u}' = (0, .. ,0,1,0, .. , 0)$ in the procedure for evaluating $\mathbf{x}'(\mathbf{u},\mathbf{u}')$ lets us evaluate the Jacobian $\mathbf{J}(\mathbf{u})$ of the mapping from $\mathbf{u}$ to $\mathbf{x}$.

The availability of the above functions for geometry and its derivatives gives direct access to kinematic capabilities. Static and inverse dynamic analysis may be built on top of the kinematic analysis by applying the principle of least work.

Let m_i be the mass of point i and let (f_i , g_i) be the external force applied to point i. Let I_j be the moment of inertia of line j and let T_j be the applied torque.

Let $\mathbf{p}$ be the generalized momentum vector and $\mathbf{f}$ the generalized force vector defined as follows:

$$\begin{aligned}
p_{2i-1} &= m_i x_i' && (1 \le i \le m)\\
p_{2i} &= m_i y_i' && (1 \le i \le m)\\
p_{2m+j} &= I_j q_j' && (1 \le j \le n)\\
f_{2i-1} &= f_i && (1 \le i \le m)\\
f_{2i} &= g_i && (1 \le i \le m)\\
f_{2m+j} &= T_j && (1 \le j \le n)
\end{aligned} \tag{12}$$

Let f_i be the applied force at point x_i and m_i its mass. At kinetostatic equilibrium the reaction force F_j in constraint j may be derived from the following equation:

$$F_j\, du_j = S(f_i - p_i').dx_i \tag{13}$$

This is an expression of the principle of least work, along with d'Alambert's principle for converting an inverse dynamics problem into a statics problem. It is also an expression of the fact that the constraints are independent.

Hence:

$$F_j = S(f_i - p_i').dx_i/du_j \tag{14}$$

Where dx_i/du_j is simply the i,j th component of the Jacobian $\mathbf{J}$.

Both $\mathbf{p}'(\mathbf{u},\mathbf{u}',\mathbf{u}'')$ and $\mathbf{J}(\mathbf{u})$ are known from the variational geometry. Hence we can compute reaction forces in the constraints which make up the variational description. This comprises a solution to the inverse dynamics problem.

The instantaneous forward dynamics problem is conveniently phrased in variational geometry terms as follows. Given a set of applied forces $\mathbf{f}$, and initial constraint values $\mathbf{u}$ and velocities $\mathbf{u}'$, find resulting constraint accelerations $\mathbf{u}''$ for a subset of constraints which are free to move in response to the unbalanced forces (the free constraints). Instantaneous solution of the forward dynamics problem can be integrated using standard ODE techniques to yield a time solution of the problem.

A solution to the forward dynamics problem may be formulated from the observation that while in dynamic equilibrium, the free constraints must transmit 0 force. Hence if constraint j is free,

$$\Sigma_i(f_i - p_i').dx_i/du_j = 0 \tag{15}$$

Now as $\mathbf{x}''$ and hence $\mathbf{p}'$ is linear in $\mathbf{u}''$ for fixed $\mathbf{u}$ and $\mathbf{u}'$, this gives a linear condition on the unknown values of $\mathbf{u}''$. As there is one such condition for each free constraint, these form a linear system of the correct size to compute the free constraint accelerations. As long as the system of constraints describing the problem is not singular, this linear system is invertible and allows us to compute the free constraint accelerations.

3. MECHANISM DESIGN WITH VARIATIONAL GEOMETRY

The previous section described, from a technical point of view, the architecture of a mechanism design software package based on Constructive Variational Geometry. In this section, we describe how such a system appears to the user. In particular, we describe the Analytix mechanism design package developed by the author.

To create a model of a mechanism in the Analytix package, the user first sketches the mechanism. Typically the sketch contains a simplified representation of the mechanism's geometry. Links are usually represented by straight lines between joints or points of force application. Centers of gravity of any links with non-negligible mass are explicitly sketched.

The user then specifies the exact geometry by adding constraints to the drawing. The constraints specify angles or distances on the drawing. For a kinematic analysis, the constraints should represent quantities whose motion is known. In practice, for a single degree of freedom mechanism, all the constraints but one represent quantities which stay fixed throughout the motion of the mechanism. The final constraint represents the driver, whose motion is known.

The Analytix system aids the constraint specification process by giving continuous feedback on whether the model is underconstrained, overconstrained, or consistently constrained.

To perform a kinematic analysis, the user specifies the velocities and accelerations of any constraints which are in motion. (In the example of the four bar linkage (figure 1), all constraints would have 0 velocity and acceleration except the driving angle. If the crank is being driven at constant angular velocity v the angle would be given a velocity v and an acceleration 0.) The user also must specify a fixed point and a fixed line.

With the constraint velocities and accelerations given, the model is able to output on demand the position, velocity, and acceleration of any point of the mechanism and the angular velocity and angular acceleration of any line.

Static analysis of a mechanism may be performed in a very natural way in a Variational Geometry system by assuming that constraints carry reactions. Hence not only the motion, but also the statics of the model are defined by the constraints. If loads are applied to a mechanism in Analytix, reaction forces are generated in the constraints. The system will sum the constraint forces which impinge on a point to give the reaction force at that point. The system will integrate the forces on a single line segment to give shear force and bending moment.

Analytix allows external forces to be applied to any point on a mechanism and external torques to be applied to any line. These forces may be constant or they may vary with geometry, time or some other parameter. Analytix also provides conformal force elements: translational and rotational spring-damper-actuators.

To perform inverse dynamics in Analytix, the user specifies the instantaneous motion of the mechanism in the same way as for kinematics, but also applies forces and specifies the mass of points representing mass centers. The user also specifies the moment of inertia of lines which represent rigid links. The user can now generate the same reaction force outputs as for the static analysis, except the inertial forces generated by the accelerating masses will be taken into consideration.

In a dynamic analysis, some constraints represent quantities whose behavior is known in advance - either quantities which remain fixed or drivers, whose motion is given. Other constraints are free to accelerate in response to unbalanced forces present in the model. To set up a dynamic model, the user specifies the fixed constraints and the input drivers in the same way as for kinematic analysis. Free constraints are given initial values and initial velocities and marked as being free to accelerate. The user then specifies a time interval over which the motion is to be integrated and an interval at which snapshots of the motion are to be taken.

The result of the dynamic analysis is a sequence of models, each of which represents a single instant of the motion. These models may be interrogated individually or as a group for any of the kinematic or reaction force outputs mentioned above. This provides the user with a high level of interactivity in the manipulation of the results of a simulation.

An advantage of a constraint based mechanism design package is the ease with which the flexibility of the software may be enhanced by integrating general purpose mathematical tools with the special purpose mechanical tools described above. For example, the Analytix program provides an equation calculator with a two way link to the geometry and mechanics of the model. Expressions generated in the calculator can be used as input forces or constraint values for the model. Conversely, geometry, velocities, accelerations or reaction forces from the model may be used in the equation calculator. In addition, a numerical root finding capability is provided.

Having the general purpose mathematical tools with full access to the mechanical model provides a powerful capability for the user to manipulate his mechanism design and is a strong advantage to the variational geometry approach.

4. EXAMPLES

In this section, we give examples of the use of the variational geometry system in mechanism design problems. Our goal in presenting these examples is both to show how the constraint based mechanism package is used in practice, and also to exemplify how the general purpose equation solution capabilities enhance the flexibility of the tool for solving real world problems.

The first example (figure 2) is in the design of a cutting tool for use with a conveyor belt. The mechanism is driven by a stepper motor at A which turns the crank AB. Line BD is constrained to slide through point C, which is fixed with respect to point A. In order to match the speed of the conveyor belt, we require the horizontal component of the velocity of D to be constant

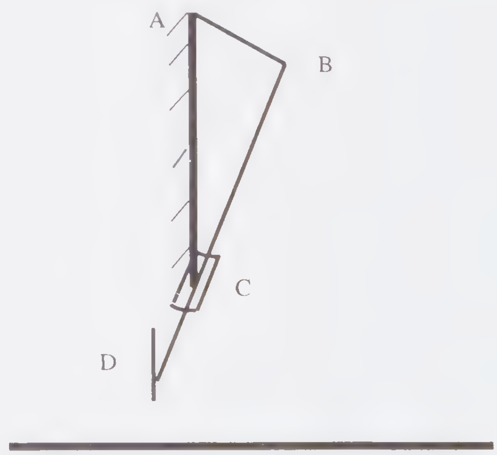

FIGURE 2: Crank AB in this mechanism is driven by a stepper motor at A. BD is constrained to pass through point C. The blade at D is required to maintain a constant horizontal velocity through the cutting phase of its motion.

through the cutting phase of the motion. The problem we wish to address is to derive a velocity profile for the stepper motor which gives a unit horizontal velocity component to D.

Figure 3 shows a variational model of this mechanism. The angle CAB is given unit angular velocity and the x component of the velocity of D may be

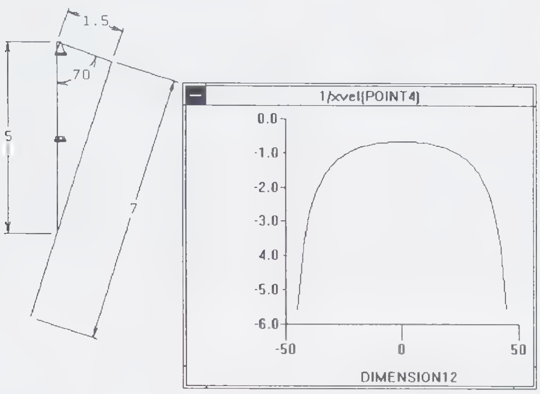

FIGURE 3: A variational geometry model of the mechanism of figure 2. A graph of the reciprocal of the horizontal velocity of the end-effector is shown for constant input angular velocity, for crank angles between -45 and 45 degrees.

output. In section 2 above we saw that, for fixed constraint values, the velocity of the geometry of our variational model is a linear function of constraint velocity. We also know from our model that 0 input angular velocity will result in 0 output velocity. Hence if xvel(D) is the horizontal component of the velocity of point D with unit input angular velocity, then the angular velocity ω which generates unit output velocity is ω = 1/xvel(D). This output function may be simply tabulated or graphed in Analytix.

Our second example (figure 4) is a parallel lift mechanism, where a mass positioned at point E is counteracted by a spring/damper between points F and G. We wish to examine the dynamic response of this mechanism when it is perturbed about its position of static equilibrium. First we need to find the position of static equilibrium.

A variational geometry model of this mechanism is shown in figure 5. The model specifies the lengths of the various members, and also specifies an angle. The angle does not represent a physical restraint but rather parametrizes the position of the lift. The static model used by the variational geometry system, however assumes the presence of a reaction torque in this angle in order to maintain equilibrium at the given angle value. In real life, however, there is no moment bearing structure at this joint, hence for static equilibrium the angle value must be such that no reaction force is present in the constraint. To find the equilibrium

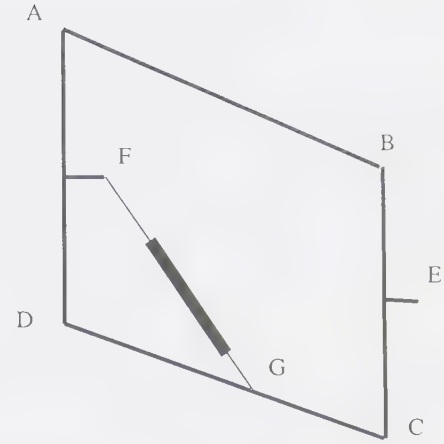

FIGURE 4: A spring between F and G supports a mass at point E on this parallel lift mechanism.

configuration, therefore, we need to find the value of the angle such that its reaction torque is zero. The root finder in Analytix may be used to find this value (figure 5).

To examine the dynamic response of this model, we specify that the angle is free to move and give it an

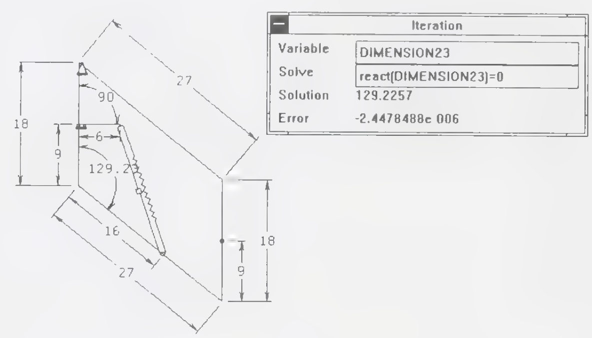

FIGURE 5: An iterative equation solver may be used to determine the angle which corresponds to static equilibrium for the mechanism of figure 4.

initial value perturbed from the equilibrium value (figure 6).

In our third example, we use the system to analyze the motion of a geared nine bar mechanism. The gear pair is modelled (figure 7) by specifying the angle of the output link as a linear function of the angle of the input link. Figure 7 shows a displacement curve for such a linkage.

The program automatically gives the torque Tout on the output link and Tin on the input link. This

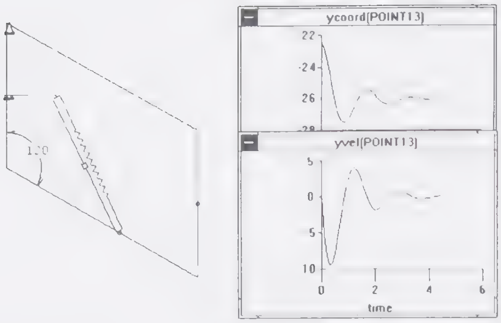

FIGURE 6: The angle of the mechanism of figure 4 is specified as a free constraint in the performance of a dynamic analysis.

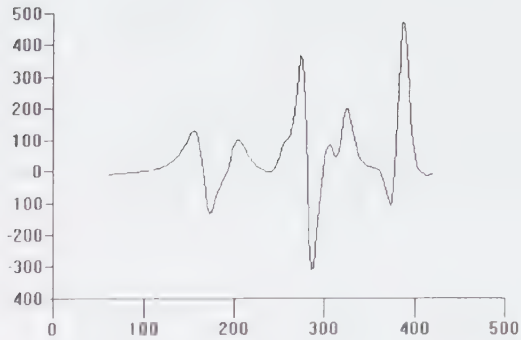

FIGURE 8: A graph of input gear torque required to move the above mechanism at a constant velocity with a point mass at the point used in the above displacement curve.

assumes a model where both links are held in equilibrium independently. In actual fact, however, the torque on the output gear is provided by a tangential force provided by the contact force between the gears. If r_{in} is the radius of the input gear and r_{out} the radius of the output gear, then this contact force is:

$$T_{out} \cdot r_{out} \tag{16}$$

The torque on the input gear must be adjusted to account for this contact force as follows:

$$T = T_{in} - T_{out} \cdot r_{out} / r_{in} \tag{17}$$

This torque is displayed in figure 8.

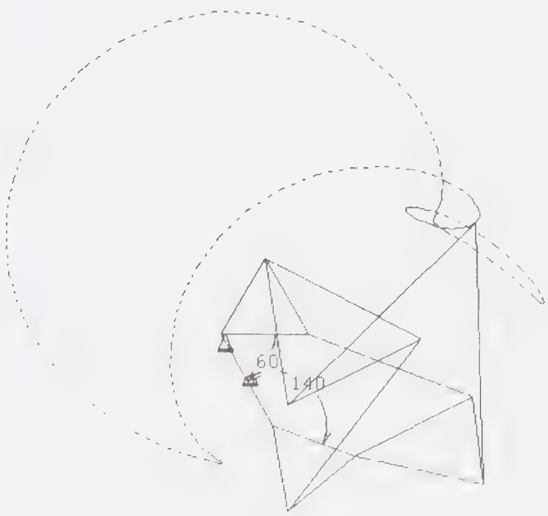

FIGURE 7: The two angles shown drive this geared nine bar linkage. Output gear angle is specified as a linear function of the input gear angle. In this model, a unit gear ratio is used.

5. DISCUSSION

Although kinematics and dynamics software has been available for well over a decade, practical mechanism design is still frequently practiced with pencil and paper, with hardware prototypes, or with the limited computer assistance of a CAD package. One reason for this is that conventional dynamics software is rather difficult to learn and non-interactive in its application. It is thus perceived as applicable only to the most complex of dynamics problems and not to the simple problems routinely addressed by the design engineer.

The objective of this work on a variational geometry based mechanism analysis package was to provide highly interactive and flexible user interface to sophisticated dynamic analysis functionality.

A variational geometry model consists of a sketch with constraint values specified on the sketch. This model of geometry corresponds closely to the natural mode in which the geometry of engineering problems is expressed, and is thus a very easy user interface for the engineer to learn. Further, it is extremely easy to modify a constraint based geometry description either by manual intervention or as part of some numerical analysis procedure. This ensures that the software is flexible enough to be of use in a wide range of real-world situations, some not within the pure domain of mechanism analysis.

In this paper, we have presented a new approach to variational geometry based mechanism design. Our method provides a core capability, within our geometry solver, of solving for first and second derivatives of the geometry. It is important to note that this solution is performed analytically, and not by means of finite differences; hence it does not suffer the inaccuracy inherent in numerical computation of derivatives. The derivative information thus obtained gives us kinematic analysis directly and dynamic analysis via the principle of least work and d'Alambert's principle. Again, this analysis is performed without resorting to iterative techniques.

Solution of the dynamic equations depends on the system of constraints not being singular. The burden of providing a non-singular parametrization of the problem is placed on the user. This has the advantage of keeping the user in close touch with the numerical process which is being performed and providing him with the tools to reparametrize the model. The disadvantage is that manual identification of singularities is not always easy.

In our examples we showed the variational geometry descriptions of several mechanism design problems. In addition we showed how equation calculation and numerical analysis procedures may be used in conjunction with the variational geometry model to solve design problems.

6. CONCLUSION

We have shown how a fully functional mechanism analysis system may be built on top of a constructive variational geometry user interface. We claim that such a system offers a unique combination of power, interactivity and flexibility. It thus has a place, alongside more conventional dynamical analysis systems in the suite of tools which may be brought to bear on the problem of mechanism design.

ACKNOWLEDGEMENT

This research was partially funded by the National Science Foundation under grant number ISI-8961200

REFERENCES

Chace, M.A., (1984) "Methods and Experience in Computer Aided Design of Large-Displacement Mechanical Systems", *Computer Aided Analysis and Optimization of Mechanical System Dynamics, (E.J. Haug ed.) 233-259*, Springer-Verlag, Berlin

Lin V, Gossard D, Light R (1981) "Variational Geometry in Computer Aided Design", *Computer Graphics 15, 171 - 177.*

Light R., Gossard D (1982) "Modification of geometric models through variational geometry", *CAD 14:209 - 214.*

Mittelstadt, W.A., D.R. Riley and A.G. Erdman (1985) "Integrated CAD of Mechanisms", *Mechanism and Machine Theory 20, 303-311.*

Todd P. (1986) "An algorithm for determining consistency and manufacturability of dimensioned drawings" *CAD86 pp 36 - 41,* Butterworths, London

Todd P. and Cherry G., (1989) "Symbolic Analysis of Planar Drawings", *Lecture Notes in Computer Science Vol 358 (P. Gianni ed.) 344-355*

Todd P. (1989) "A k-tree generalization that characterizes consistency of dimensioned engineering drawings", *SIAM Journal of Discrete Mathematics 2:255-261.*