A System or Automated Deduction in Engineering Mechanics

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Abstract. A system is presented for automated formula discovery in engineering mechanics built on a Lagrangian formulation where geometric constraints are treated as physical constraints. While the architecture of the system is described, the main focus of this paper is to highlight the interplay between architecture and user interface in generating formulas for mechanical problems. With this in mind , a number of examples are presented in the kinematics, statics and dynamics of simple mechanisms.

Keywords: Mechanics · Automated deduction

1 Introduction

The techniques of automated deduction in geometry have been applied to mechanics in a number of different settings. In [16] Wu extends his method to differential geomery and to the derivation Newton's and Kepler's Laws of planetary motion. This work is extended by Chou and Gao [2] and Wang [15]. In [3], the theorem prover ISABELLE, along with nonstandard analysis is applied to the automated derivation of the theorems in Newton's Principia. In addition to celestial mechanics, Chou and Gao apply their methods to more mundane problems in plane kinematics [1]. In [5], Groebner Bases are applied to solve nonlinear constraint problems emerging from the interplay between geometry and mechanics in the statics of trusses. The question of which beams bear zero loads in a specific problem, and always bear zero loads is addressed using the automated deduction system OTTER in [4].

In this paper, we describe a system which layers Lagrangian mechanics on top of a geometric constraint model to generate formulas in engineering mechanics. The system is designed to be general purpose within its domain of application and to give symbolic solutions for kinematic, static, kinetostatic and dynamic problems involving simple

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machines of the sort analyzed in engineering texts. An engineering mechanics model has at its root a geometry model. Our mechanics system is built on top of a constraint based geometry system *Geometry Expressions* and intertwines the constraint description with the mechanics model. In a numerical context, this approach is embedded in the product *Analytix* and described in [10]. The architecture for a symbolic system is discussed in [11]. In this paper, we present a fully realized symbolic mechanics system, discuss user interface and give usage examples. We thus explore its strengths and weaknesses, hopefully elucidating the relationship between the constraint model and the mechanics.

A fundamental characteristic of the architecture is the treatment of constraints as being “load bearing”. That is, a reaction force is computed for each constraint. While this is the familiar Lagrangian formulation of mechanics, it necessitates a shift in perspective for engineers used to thinking of objects as bearing loads, rather than constraints. However it admits a very parsimonious model description, allowing the essence of an engineering problem to be expressed in a way which can generate succinct symbolic forms of the solution.

The system, *Mechanical Expressions [14]* uses a constraint based geometry description and layers on top of this kinematic elements. Velocities and accelerations may be assigned to the constraints, and the resulting velocities and accelerations of geometric elements may be measured. The underlying geometry system provides a map G from symbolic constraint values to the symbolic Cartesian coordinates of the points in the model. Differentiation of this mapping provides a kinematic analysis of the model. *Mechanical Expressions* runs on Windows or the Mac.

The Euler Lagrange equations [7] admit a simple expression in the form of the principle of virtual work. This states that at static equilibrium the work done to move the model incrementally is zero. This is an expression of the fact that the model is at an energy minimum. For each constraint, an infinitesimal change in its value will result in an infinitesimal change in the location of all the points and lines in the model. This is embodied in the Jacobian of G. The virtual work is the sum of the applied forces multiplied by the infinitesimal change in location along the axis of the applied force. We define the reaction force in a constraint to be the force which balances out the incremental virtual work done against the applied force elements.

Interpreting the constraint reaction force is part of the skill of model building in the system. For a simple truss (fig. 1) each distance constraint corresponds to the presence of a physical beam, and the reaction force in the constraint corresponds to the internal tension or compression in the beam. A model of a slider crank mechanism (fig. 2) has two distance constraints and an angle constraint. The distances correspond to physical members (the crank and the connecting rod) and would be expected to remain constant. The angle's value would change during the motion of the mechanism. The reaction force in the angle (actually a torque) would correspond to the torque required of the motor.

Inverse dynamics, or kinetostatics require the addition of mass elements and the provision of constraint values with velocities and accelerations (or the ability to describe them as time dependent functions). Given these constraint velocities and accelerations, the system can compute the velocity and acceleration of any point in the model and the angular velocity and acceleration of any line by differentiation. Inertial force elements are added to correspond to accelerating mass elements. Reaction forces may be computed as in the static case (fig. 3). Masses of bodies are represented by a point mass at the center of gravity along with a moment of inertia about the Center of grav-

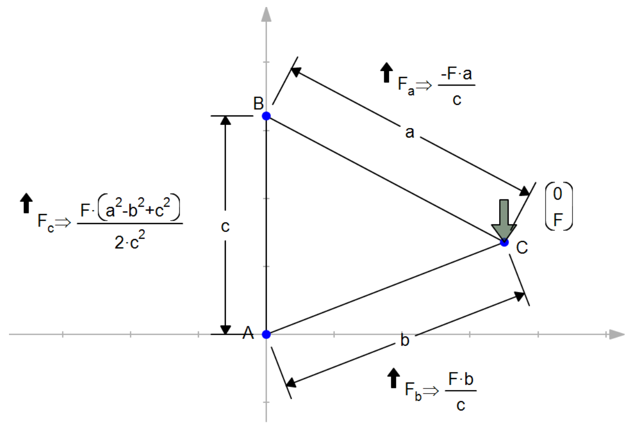

Fig. 1 A triangular truss with force applied at B. Constraint forces are displayed for the three length constraints.

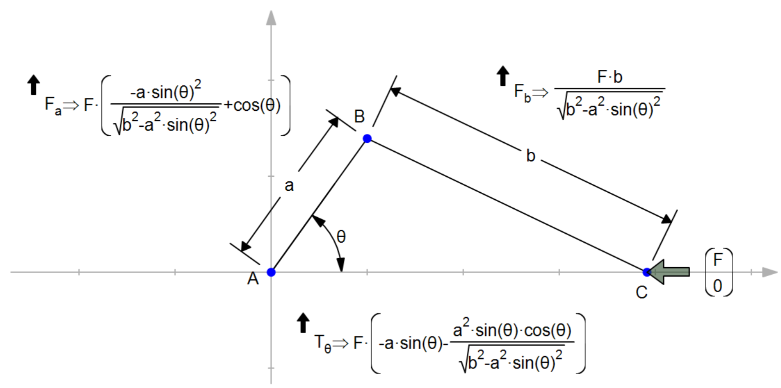

Fig. 2 A crank slider mechanism. External force is applied at C and constraint forces are displayed in the length constraints a and b and the angle constraint θ

ity attributed to any line which moves with the body. Mass elements may or may not experience a gravitational force, depending on whether the figure is specified as lying in a vertical, horizontal, or inclined plane.

In inverse dynamics, forces are derived for a given motion. By contrast a dynamic model derives motion for a given force. The specification of a dynamic model involves identifying one ore more constraints which are free to accelerate in response to unbalanced forces. An acceleration of a constraint will be mapped into accelerations of the masses in the figure, and hence change the reaction forces in the various constraints. The accelerations of the free constraints need to be such that the reaction forces in these constraints are zero. This condition can be expressed as a linear system for the free constraint accelerations.

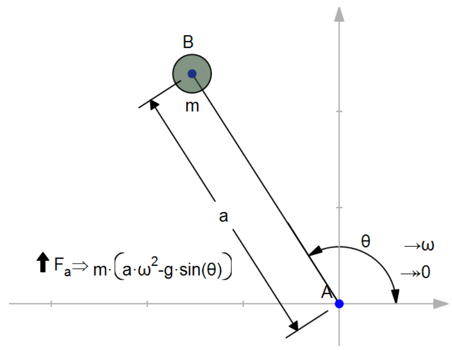

Fig. 3 AB rotates in a vertical plane with angular velocity ω. Reaction force in the length constraint is shown. One term is due to gravity acting on the mass, the other term is a centripetal pseudoforce introduced by the accelerating mass.

2 Mechanics Architecture

Geometry Expressions [12] is a dynamic geometry system with two defining characteristics. While it has a construction based geometry engine, it has a constraint based user interface along with a graph algorithm [9] which maps from a constraint based description to a construction sequence. Constraint arrangements which do not admit a construction sequence are not permitted. A second characteristic is that constructions are performed symbolically rather than numerically. Hence *Geometry Expressions* may be viewed as providing a map from constraint values to coordinate locations.

Let $q_1, \ldots, q_m$ be the (symbolic) constraint values. Let $(x_1, y_1), \ldots, (x_n, y_n)$ be the cartesian coordinate locations of the points of the model and $\theta_1, \ldots, \theta_k$ the angles of the lines.

Let

$$\mathbf{q} = (q_1, \ldots, q_m)$$

and

$$\mathbf{x} = (x_1, y_1, \ldots, x_n, y_n, \theta_1, \ldots, \theta_k)$$

Geometry Expressions provides the function.

$$\mathbf{x}(\mathbf{q})$$

As the function is symbolic, its Jacobian:

$$J_{ij} = \frac{\partial x_i}{\partial q_j}$$

may be readily computed by differentiation. This Jacobian along with the Hessians:

$$H_{ijk} = \frac{\partial^2 x_i}{\partial q_j \partial q_k}$$

are central to the development of a symbolic mechanics capability.

Our symbolic mechanics capabilities are layered, with each layer requiring additional mechanical information on top of the pure geometry. The sequence is Kinematics, Statics, Kinetostatics, Dynamics.

2.1 Kinematics

The core kinematics problem is this: given constraint velocities and accelerations $\dot{\mathbf{q}}, \ddot{\mathbf{q}}$ to find geometry velocities and accelerations $\dot{\mathbf{x}}, \ddot{\mathbf{x}}$.

These can be evaluated directly using the Jacobian and Hessians.

$$\dot{x}_i = \sum_j J_{ij} \dot{q}_j$$

$$\ddot{x}_i = \sum_j \sum_k H_{ijk} \dot{q}_j \dot{q}_k + \sum_j J_{ij} \ddot{q}_j$$

A secondary problem is to compute the velocity and acceleration of any geometric measurement, such as a distance or an angle.

For such a problem, assume the measurement be expressed as a function $\mu(\mathbf{x})$. Then

$$\dot{\mu} = \sum_i \frac{\partial \mu}{\partial x_i} \dot{x}_i$$

$$\ddot{\mu} = \sum_j \sum_i \frac{\partial^2 \mu}{\partial x_i \partial x_j} \dot{x}_i + \sum_i \frac{\partial \mu}{\partial x_i} \ddot{x}_i$$

User Interface

Kinematics is implemented in *Mechanical Expressions* by providing velocity and acceleration input elements which may be attached to any constraint in the system. An output velocity and acceleration element may be attached to any point or line in the drawing. In the case of a point, a velocity and acceleration vector are returned. In the case of a line, an angular velocity and acceleration is returned.

An output kinematic element may also be added to any measurement, and will display the velocity and acceleration of that element. In the case of a point, these will be vector quantities. In the case of a line they will be angular velocity and acceleration. Figure 4 illustrates the solution of a kinematics problem.

2.2 Statics

Given external forces (f_i, g_i) applied to the points in the drawing, and torques T_j applied to the lines , let $\mathbf{f} = (f_1, g_1, \ldots, f_n, g_n, T_1, \ldots T_k)$. We define the generalized constraint force [8]

$$Q_j = -\sum_i f_i J_{ij}$$

By construction, we see that

$$\sum_j Q_j \delta q_j + \sum_i f_i \delta x_i = 0$$

Which can be thought of as an expression of the Principle of Virtual Work [8]

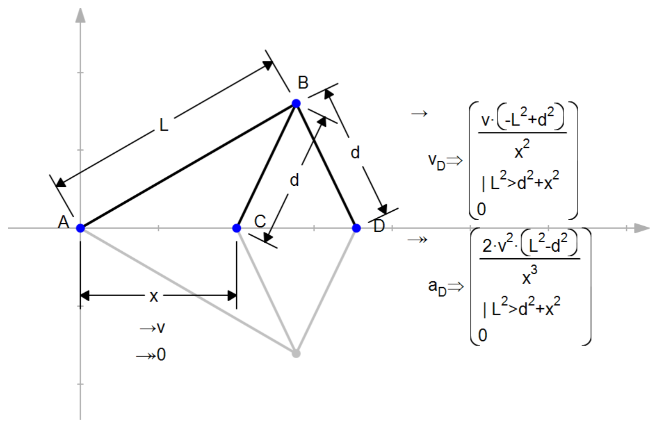

Fig. 4 Input kinematics element specifies velocity and acceleration for the length constraint x. Output kinematic element measures cartesian velocity and acceleration of point D.

In the case of a distance or length constraint, the constraint force represents the internal force in whatever element maintains the constraint. If the element is a rigid body, then this is the tension or compression force exerted to maintain the body's rigidity. If the element is a linear actuator, then this is the force expended by the actuator.

In the case of an angle constraint, the generalized constraint force represents the torque required to maintain that angle.

User Interface

Applied force elements are provided, which may be attached to points in the diagram. Examples of applied forces may be seen in figures 1and 2. In addition, spring-damper-actuator elements may be attached to pairs of points, the effect of which is to apply an equal and opposite force along the line of action of the element to each of the end points. The magnitude of the force is computed from the parameters of the element. For a spring with free length L, spring constant k and end points (x_0, y_0) and (x_1, y_1) The magnitude of the force is:

$$F = k \cdot \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} - L$$

For a damper with damping coefficient c, the magnitude of the force is

$$F = c \cdot \frac{(\dot{x}_1 - \dot{x}_0) \cdot (x_1 - x_0) + (\dot{y}_1 - \dot{y}_0) \cdot (y_1 - y_0)}{\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}}$$

For an actuator, the magnitude of the force is given directly by the user. Examples of such elements may be seen in fig. 6 and fig. 9. These force elements have torque analogues for lines and line pairs: applied torque attached to a line, angular spring/damper/actuators between pairs of lines. Such an element is used in fig. 13.

The principal static output is the reaction force element. This can be attached to any constraint in the model and gives the generalized constraint force. For linear constraints, this can be interpreted as an internal force. For angular constraints, it can be interpreted as a torque.

2.3 Kinetostatics

The kinetostatic, or inverse dynamic problem is essentially solving $F = m \cdot a$ for F given a. It is equivalent to the static problem with the addition of inertial forces corresponding to accelerating masses. Given masses m_i applied to the points in the drawing, and moments of inertia I_j applied to the lines, let

$$\mathbf{p} = (m_1\dot{x}_1, m_1\dot{y}_1, \ldots, m_n\dot{x}_n, m_n\dot{y}_n, I_1\dot{\theta}_1, \ldots, I_k\dot{\theta}_k)$$

Inertial force quantities $\dot{\mathbf{p}}$ are added to the derivation of the generalized constraint forces.

$$Q_j = -\sum_i (f_i - \dot{p}_i)\, J_{ij}$$

In addition to the inertial force, the mass element may experience a gravitational force of $m \cdot g \cdot sin\theta$ where θ is the angle of the model's plane to the horizontal and g is acceleration due to gravity.

Let

$$\mathbf{G} = (0, -m_1 g, \ldots, 0, -m_n g, 0, \ldots, 0)$$

then adding gravity, the generalized constraint forces are

$$Q_j = -\sum_i (f_i + G_i - \dot{p}_i)\, J_{ij}$$

User Interface

To enable the creation of kinetostatic models, *Mechanical Expressions* provides two mass elements. The point mass may be attached to a point of the model. The user is also able to specify the angle of the plane of the model to horizontal. If this angle is non-zero, gravitational forces are applied to any point mass elements. Figure 11 shows the use of both a point mass positioned at the center of gravity, and a moment of inertia applied to a line to model the mass properties of a uniform beam.

The output mechanical elements for kinetostatics are, as for statics, the generalized constraint reaction forces.

2.4 Dynamics

If kinetostatics is solving $F = m{\cdot}a$ for F given a, then dynamics is solving $F = m{\cdot}a$ for a given F. Underlying the previous forms of mechanical analysis is the assumption that the value, velocity and acceleration of the constraints are given, and hence the motion is known. If the motion is unknown and the model is fully constrained, there must be one or more constraints whose accelerations are to be derived from a consideration of the applied forces. *Mechanical Expressions* discriminates between two types of constraints: those which are held constant or in prescribed motion, and those which are free to accelerate. Constraints which are held constant typically correspond to rigid bodies, while constraints with prescribed motion typically represent motors or actuators. Constraints which are free to accelerate are geometrical but not structural.

They define the parametrization of the model but not its physics. It is this provision of parametrization which motivates the apparently oxymoronic concept of a constraint which is free to change.

The vector $\dot{p}$, the derivative of the momentum can be expressed in terms of the constraints as

$$\dot{p} = m_i \ddot{x}_i = m_i \left(\sum_j \sum_k H_{ijk} \dot{q}_j \dot{q}_k + \sum_j J_{ij} \ddot{q}_j \right)$$

If we let Q^*_kbe the reaction force in free constraint k where the free constraints have 0 acceleration. Let Q_k be the reaction force with free accelerations $\alpha_1, \ldots, \alpha_r$

$$Q_k - Q^*_k = \sum_i \dot{p}_i J_{ik}$$

$$= \sum_i m_i \left(\sum_j J_{ij} \alpha_j \right) J_{ik}$$

$$= \sum_j \left(\sum_i m_i J_{ij} J_{ik} \right) \alpha_j$$

Let

$$M_{kj} = \sum_i m_i J_{ij} J_{ik}$$

Then

$$Q_k - Q^*_k = \sum_j M_{kj} \alpha_j$$

This is a linear equation in $\alpha_1, \ldots, \alpha_r$. At dynamic equilibrium, we require the reaction force in the free constraints to be 0. Hence $Q_k - Q^*_k = -Q^*_k$and we have the following linear system for the free accelerations:

$$\sum_j M_{kj} \alpha_j = -Q^*_k$$

Example

To illustrate how this dynamic solution works in practice, we use the kinetostatic features of *Mechanical Expressions* to derive reaction forces and solve the linear system explicitly. This is handled automatically by the software in the computation of resultant accelerations.

Figure 5 shows a model of a pendulum comprising equal masses at points A and B. Rather than being fixed, point A is free to slide along the x-axis. The model is constrained by the length AB, which will remain constant, by the location x of point A on the x-axis (this is the x coordinate), and by the angle θ between AB and the x-axis.

Constraint x is given velocity v and acceleration a. Angle θ is given velocity ω and acceleration α. The figure shows reaction forces computed by *Mechanical Expressions* for these two constraints:

$$F_x = m \cdot \left(-2a + \alpha \cdot sin\theta + \omega^2 cos\theta \right)$$

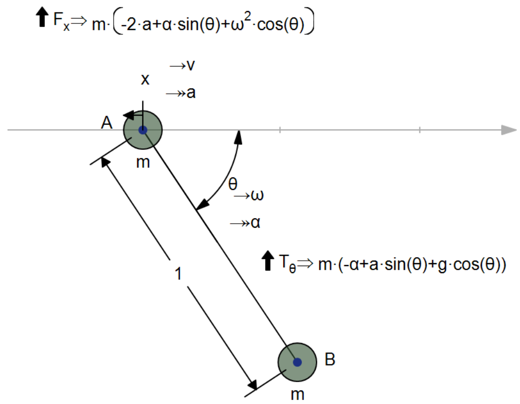

Fig. 5 Mass A is free to slide along the x-axis. AB acts as a pendulum.

$$T_\theta = m \cdot (-\alpha + a \cdot sin\theta + g \cdot cos\theta)$$

At dynamic equilibrium the accelerations a and α are such that these reaction forces are zero. This sets up the following linear system for the accelerations.

$$\begin{pmatrix} -2 & sin\theta \\ sin\theta & -1 \end{pmatrix} \begin{pmatrix} a \\ \alpha \end{pmatrix} = \begin{pmatrix} -\omega^2 cos\theta \\ -g \cdot cos\theta \end{pmatrix}$$

Whose solution is

$$a = \frac{(\omega^2 + g \cdot sin\theta) \cdot cos\theta}{2 - sin^2\theta}$$

$$\alpha = \frac{(2g + \omega^2 sin^2\theta) \cdot cos\theta}{2 - sin^2\theta}$$

User Interface

The additional input capability required for dynamic analysis is the ability to specify which of the model constraints are free to accelerate. The primary output element is the resultant acceleration, which may be measured for any of the free constraints. For example, in figure 10 angle θ is specified as free to accelerate, and its instantaneous acceleration displayed.

The free constraint accelerations, expressed in terms of constraint values and their velocities constitute a set of r second order differential equations. These can be expanded to a set of $2r$ first order differential equations by adding equations for the constraint velocities. This is a convenient form for export to mathematics systems for numerical or symbolic solution. For example, the differential equations generated in figure 5 are as follows.

$$
\begin{aligned}
w5'(\tau) &= w6(\tau) \\
w6'(\tau) &= \frac{\left(2 \cdot g + sin(w5(\tau)) \cdot w6(\tau)^2\right) \cdot cos(w5(\tau))}{1 + cos(w5(\tau))^2} \\
w7'(\tau) &= w8(\tau) \\
w8'(\tau) &= \frac{\left(g \cdot sin(w5(\tau)) + w6(\tau)^2\right) \cdot cos(w5(\tau))}{1 + cos(w5(\tau))^2} \\
w5(0) &= \theta \\
w6(0) &= \omega \\
w7(0) &= x \\
w8(0) &= v
\end{aligned}
$$

Note that new functions of time are introduced for the the constraint values and their derivatives. Displayed constraint values and velocities are used as initial conditions. The equations may be copied from *Mechanical Expressions* in a form specific to one of a number of mathematical systems. Once in the mathematics system, numerical solution of the system may readily be accomplished.

2.5 Body Mass

A body's mass properties may be modeled parsimoniously by creating a point at its center of gravity and at least one line which moves with the body. The body's mass is added to the center of gravity point. The body's moment of inertia (about the C of G) is added to any line which rotates with the model. For example, in fig 11 the mass properties of a uniform beam are modelled with a line segment constrained to have length L. A point mass m is placed at its center, while a moment of

$$\frac{mL^2}{12}$$

is added to the line.

3 Geometric Constraints and Mechanical Constraints

A characteristic of the system as described above is that the constraints which define the geometry of the model, and hence the form of all the output expressions must also be mechanical constraints: that is they should be quantities which are kept constant or whose motion is prescribed. This permits succinct statements of mechanical problems, at a cost of limiting flexibility. In addition to specifying the underlying geometry, the geometric constraints also define the coordinate system in which the mechanical model is solved. If a mass is constrained to lie at coordinates (x, y), and that constraint is made free to accelerate, then its equation of motion will be derived in cartesian coordinates. If, on the other hand, the distance to the origin is constrained to be r and the angle which a line joining the point to the origin is constrained to be θ, and both these constraints are free to accelerate, then the equation of motion will be derived in polar coordinates. In fig. 12, the mass is constrained to lie on an arbitrary curve, and the mechanics is solved in terms of parametric coordinates on that curve.

In many problems, it is convenient to parametrize the problem using geometric constraints which do indeed correspond to mechanical constraints. For some problems,

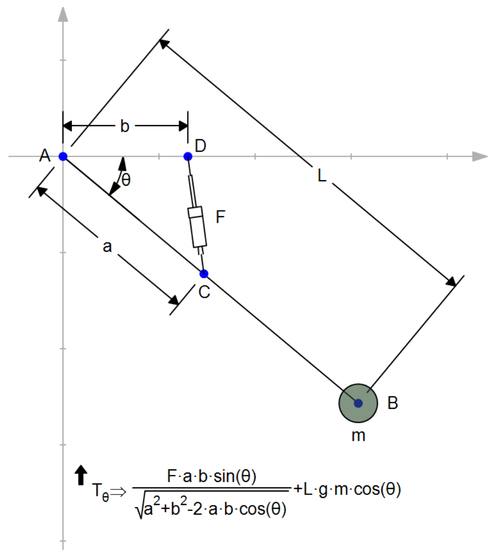

Fig. 6 Actuator CD exerts force F to hold up mass m at D. Angle θ is a geometric constraint and should bear no load. Hence the value of F should be such that $T_\theta = 0$.

however, the user may wish to parametrize the problem in terms of a geometric constraint which is not a mechanical constraint. For example, in fig. 6, the user wants an expression for the force F in the actuator as a function of the angle θ. He could have constrained the length of the actuator rather than the angle, but then his results would have been obtained in terms of this length. In fig. 6 an actuator with force F has been added and the constraint force in angle θ computed. The value of F as a function of θ may be calculated by setting the constraint force (torque) to be 0 and solving the resulting linear equation.

Let a mechanical constraint be a scalar function $\psi(\mathbf{x})$.

In general, let's assume that we have r mechanical constraints $(\psi_1, \ldots, \psi_r)$, and r geometric constraints $(q_1, \ldots, q_r)$, which are not load bearing.

Then let Ψ_i be the constraint force in mechanical constraint ψ_i and Q_j be the constraint force in geometric constraint q_j, then

$$Q_j = -\sum_i f_i J_{ij} - \sum_k \Psi_k \frac{\partial \psi_k}{\partial q_j}$$

This linear system may be solved for $\mathbf{\Psi}$ such that $\mathbf{Q}$ is zero.

From a user interface perspective, there is a need here to create three classes of constraints: those which are geometric only, those which are mechanical only, and those which are both. This distinction is not in the current system, and such problems must be handled manually.

Limitations due to Constructibility Criterion

Fundamental to the symbolic geometry system *Geometry Expressions* is the necessity of converting the constraint model into a sequential construction sequence. The con-

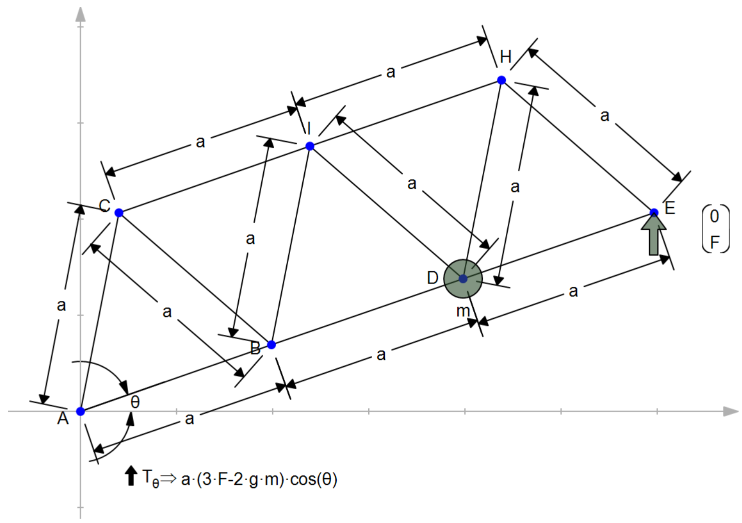

Fig. 7 The angle θ at which the bridge is cantilevered is a geometric constraint and should bear no load.

straints, in a sense are simply a User Interface device on top of a dynamic geometry construction system. The benefit of this approach is that it avoids situations which demand numerical solution. A disadvantage is that there are mechanical problems which cannot be expressed in such a constructive fashion.

For example, fig 7 shows a model of a simple truss bridge. As displayed in the model it is cantilevered at an angle of θ to the horizontal. However, this does not reflect the true mechanics of the situation, we wish the right end of the bridge, point E, to be supported by the axis. However, if we replace the angle constraint with an incidence constraint between E and the axis, the geometry becomes unconstructible from the constraints. A possible approach to this particular problem in the geometry system might be to construct the bridge at an arbitrary angle, find the location of E, then rotate by the appropriate angle such that E lies on the axis. Such an approach, implemented at the geometry level leads to unacceptable intermediate expression swell when mechanical analysis is layered on top of it.

An alternative approach is to use geometric constraints which preserve the model's constructibility (that is, we preserve the angle θ in fig 7). However, we add force elements which recapture the correct mechanical context. In the bridge example, this approach is illustrated by the addition of a vertical force F at point E. F should be set such that the constraint force in angle θ is zero. In the figure, this is done manually. However, the implementation could be automated by the addition to the system of constraints which are only geometrical, and other constraints which are only mechanical.

4 Forms of Output

Measurements of mechanical quantities are by default returned as symbolic expressions whose indeterminates are the indeterminates which are present in the constraint values and in the mechanical inputs, such as velocities and accelerations, applied forces etc. For models of even mild complexity, these symbolic outputs cna be overwhelmingly large. It is important, therefore, to provide a number of alternative forms which allow the system to give usable output as diagram complexity increases.

At the simplest level, the system has numeric values assigned to all the indeterminates, along with a user interface which allows the numeric values to be modified. Any mechanical measurement may be displayed in numeric form, where the indeterminates are replaced by their numeric values and the output expression numerically evaluated.

Intermediate to the numeric and full symbolic expressions, the system will present a result in the form of a Taylor series expanded about the current numerical values of the indeterminates. The user is able to specify the order of the Taylor Series.

For example, in figure 2 if the crank length is specified as 1 (rather than a), and the connector length 2 (rather than b), then the symbolic result for the torque in angle θ is

$$-F \cdot \left(-sin(\theta) - \frac{sin(\theta) \cdot cos(\theta)}{\sqrt{3 + cos(\theta)^2}} \right)$$

If the variable θ has numeric value 0.95, and F has numeric value 1.0, then the numeric output for the torque is

$$1.072375$$

While the 2nd order Taylor Series output is

$$-0.8613836 + 0.6530017 \cdot F + 1.813439 \cdot \theta + 0.4414461 \cdot F \cdot \theta - 0.9544417 \cdot \theta^2$$

Another useful output is in the form of source code, either code snippets, which may be embeded in the user's programs or spreadsheets, or entire JavaScript apps which allow the user to present results as interactive web object, embedding the symbolic solution in an exploratory interface (while hiding the actual symbolics.) Examples of these apps may be seen on the website [13].

5 Examples

We give a number of example problems to illustrate the different mechanical features and some techniques of using the software.

Example 1 Let ABC be a double pendulum supported at A and with equal masses at B and C. Let AB=BC. Let point C be constrained to lie on an inclined plane at angle θ to the horizontal. Find the equilibrium position of the pendulum.

To express this problem in *Mechanical Expressions*, a line is drawn passing through the origin and constrained to be at angle theta with the x axis. Point A is located at the origin, line segments AB and BC are drawn, and C is constrained to lie on the sloping line. AB and BC are constrained to have length a, and masses m are placed at B and C. Finally the angle between AB and the sloping line is constrained to be the angle ϕ (fig. 8).

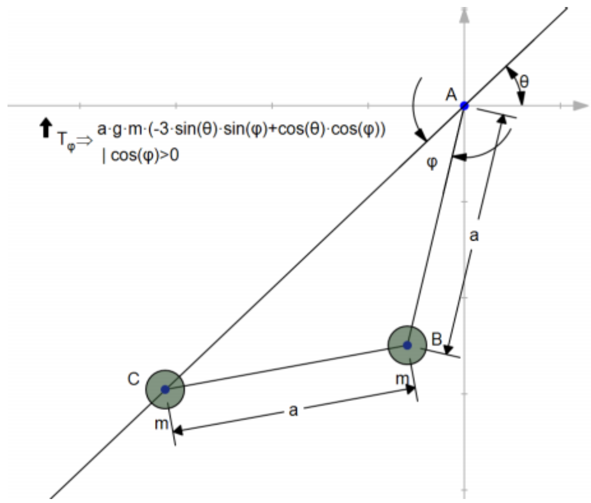

Fig. 8 Torque in the angle between AB and the inclined line is calculated. Equating this to zero finds the equilibrium position.

We can ask *Mechanical Expressions* for the torque in this angle required to maintain static equilibrium with the given geometry, yielding the expression:

$$a \cdot g \cdot m \cdot (-3 \cdot sin(\theta) \cdot sin(\phi) + cos(\theta) \cdot cos(\phi))$$

If there is nothing in place to exert the requisite torque, the system is in static equilibrium only if this reaction torque is zero. Equating the above to 0 yields the following equation for ϕ:

$$tan(\phi) = \frac{1}{3 \cdot tan(\theta)}$$

Example 2 Find a location for a zero-free-length spring to counterbalance an oven door throughout its range of motion

A zero-free-length spring is a convenient idealization of a slack spring (think of a “slinky” toy) with negligible free length. It can also be realized with a finite free-length spring by putting the spring in an enclosure, and hinging the enclosure at the natural length of the spring. Figure 2 shows a *Mechanical Expressions* model of such a door. Point B represents the center of gravity of the door, while point A, at the origin represents the door hinge. A spring with rate k, and free length 0 has one end attached to the door at distance b from the hinge, while the other end is located at coordinate location (u, v). The angle between the door and the x-axs is constrained to be θ.

In the *Mechanical Expressions* model, reaction forces in the distance constraints a, b, in the coordinate constraints (u, v), and a reaction torque in the angle θ balance the force exerted by gravity acting on the mass m and by the spring. We are particularly interested in the torque in angle θ as this does not correspond to any structural element, but reflects some externally applied force used to hold the door in a desired position. For the spring to counterbalance the door exactly, this torque should be zero.

Figure 9 displays this torque:

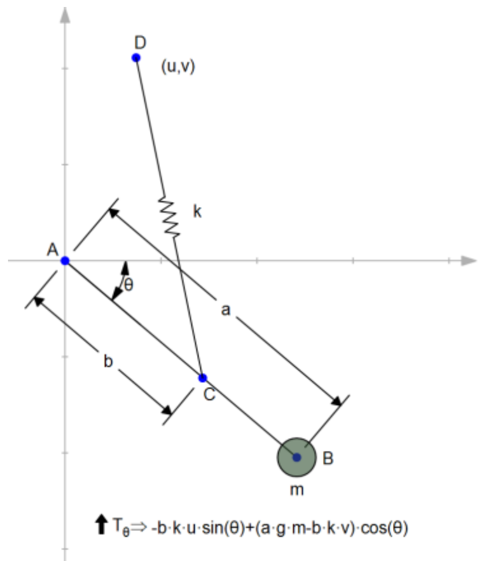

Fig. 9 Reaction torque in the angle θ is displayed. Point B is the center of gravity of the door, whose mass is m.

$$-b \cdot k \cdot u \cdot sin\theta + (a \cdot g \cdot m - b \cdot k \cdot v) \cdot cos\theta$$

This expression is zero for any θ if $u = 0$ and $b \cdot k \cdot v = a \cdot g \cdot m$. For given k and m, the designer has the freedom to choose b or v and the other is determined.

Example 3 A beam of length L has unequal masses M and m at its ends. Ignoring the mass of the beam, where would a fulcrum be located so that the angular acceleration under gravity is maximized.

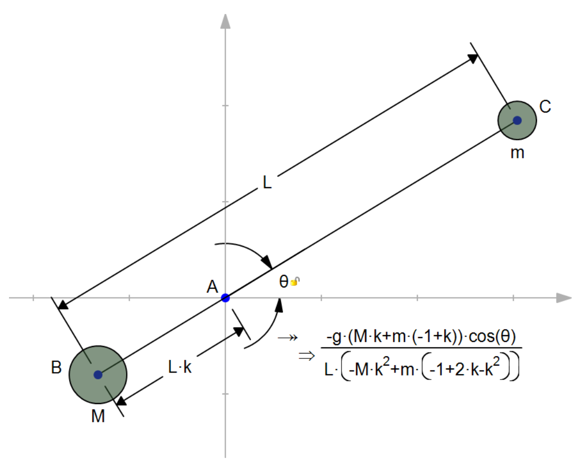

Fig. 10 Acceleration in the angle θ is displayed

In fig. 10, the fulcrum is constrained to lie proportion k along the beam, whose length and angle to the x-axis are prescribed. The acceleration of this angle may be examined.

$$\frac{-g \cdot (M \cdot k + m \cdot (-1 + k)) \cdot cos(\theta)}{L \cdot (-M \cdot k^2 + m \cdot (-1 + 2 \cdot k - k^2))}$$

To find a maximum acceleration, we need to differentiate the expression and solve. This can be done by copying from *Mechanical Expressions* into an algebra system and manipulating the result there. In this case we obtain the following solution for k.

$$k = \frac{m - \sqrt{mM}}{M + m}$$

Replacing k in *Mechanical Expressions* by this value yields the following for the acceleration:

$$\frac{-g \cdot \sqrt{M \cdot m} \cdot (M + m) \cdot cos(\theta)}{2 \cdot L \cdot M \cdot m}$$

We note that this acceleration is proportional to the ratio of the arithmetic mean and the geometric mean of m and M.

Example 4 An exhibit in a science classroom consists of a yardstick hinged at one end. At the other end are two small cups, and a ball sits in the outer cup. A wedge is placed under the yardstick so that it sits at a particular angle. when the wedge is removed, the yardstick falls, and the ball is found to have moved from the outer cup to the inner cup. Why? What restrictions are there on the angle of the wedge?

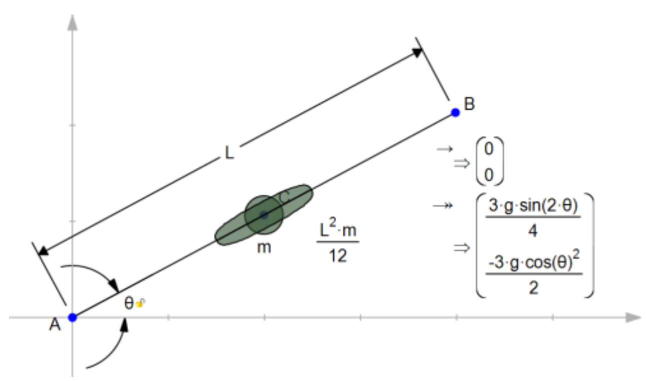

Fig. 11 Uniform beam AB is hinged at B. We measure the acceleration of point B as the beam falls under gravity.

The ruler can be modeled by a line, with one end fixed at the origin. The length of the line is constrained to be L Its center of mass is at the midpoint. A mass m is placed there. The moment of inertia about the center of mass of a uniform beam is

$$\frac{mL^2}{12}$$

This inertia is given to the line. The angle between the line and the x-axis constrained to be θ.

This angle is specified as being free to accelerate. The vertical component of the acceleration of point B is shown to be:

$$\frac{-3 \cdot g \cdot cos(\theta)^2}{2}$$

Which is greater than g so long as

$$cos(\theta) > \sqrt{\frac{2}{3}}$$

or θ less than about 35 degrees.

As the ball falls with acceleration g, under these circumstances the end of the ruler accelerates faster than the ball, allowing it to escape its cup and drop into the other, strategically positioned cup.

Example 5 A wire has the shape of the curve $(x(t), y(t))$. A bead slides on this wire under the influence of gravity. If the bead is at parametric location t, and has parametric velocity v, what is its parametric acceleration under gravity? Assume gravity operates in the negative y-direction.

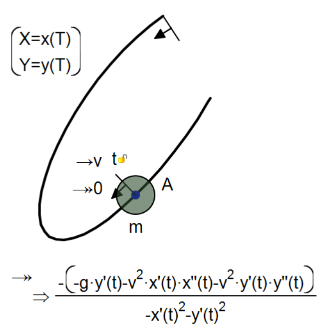

Fig. 12 A mass is located at A which is constrained to be at parametric location t on the curve $(x(t), g(t))$. It is given parametric velocity v. The resulting parametric acceleration is displayed.

Figure 12 shows a model for this problem. Point A is constrained by a parametric location constraint, whose value is specified to be t and whose velocity is specified to be v. The constraint is set to be free to accelerate, and its resultant acceleration is output.

$$a = \frac{-\left(-g \cdot y'(t) - v^2 \cdot x'(t) \cdot x''(t) - v^2 \cdot y'(t) \cdot y''(t)\right)}{-x'(t)^2 - y'(t)^2}$$

Note that v is the parametric velocity of the bead, while x',y', x'', y'' are functions describing the slope and curvature of the curve.

Example 6 Using Force to Solve Geometry Problems

In *the Mathematical Mechanic* [6], Mark Levi inverts the normal order of things, using physics to solve mathematical problems rather than the other way round. His method is to dream up a physical thought experiment whose solution coincides with the solution of the posed mathematical problem. Physical considerations can lead to slick proofs both of the thought experiment and of the corresponding mathematical problem.

In this section we illustrate how Levi's methods can be combined with *Mechanical Expressions* to solve a geometry problem.

The Drive in Movie Problem [6] (otherwise known as the rugby kick problem or the art gallery problem) is this. How far back should you park to get the best view of the movie screen at a drive-in movie?

Mathematically, we want to maximize the angle subtended by the movie screen. Levi's model for this is to take the two sight lines (AC and BC in figure 13) and add a constant torque angular actuator between them. The potential energy in such an actuator is proportional to torque times angle. As the torque is constant an angle stationary point will correspond to an energy stationary point. The physical problem of finding a static equilibrium for the model aligns with the mathematical problem of finding a maximal angle.

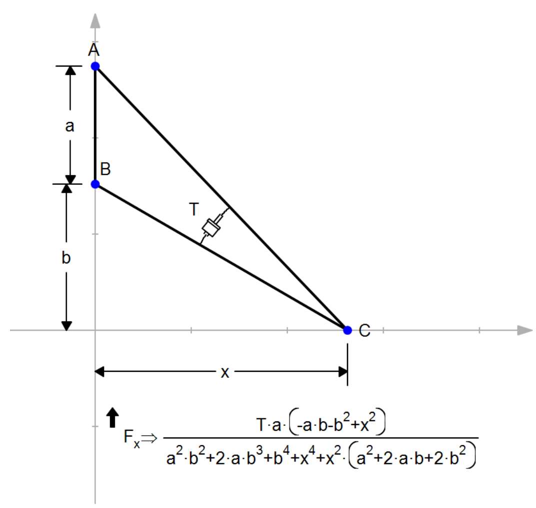

Fig. 13 A model for the drive-in movie problem

In Figure 13, the force in the distance constraint x is displayed.

$$\frac{T \cdot a \cdot \left(-a \cdot b - b^2 + x^2\right)}{a^2 \cdot b^2 + 2 \cdot a \cdot b^3 + b^4 + x^4 + x^2 \cdot (a^2 + 2 \cdot a \cdot b + 2 \cdot b^2)}$$

At equilibrium this force must be zero, leading, by inspection, to the solution:

$$x = \sqrt{b^2 + a \cdot b}$$

We admit that deploying the heavy artillery of an automated mechanics system is somewhat at odds with the spirit of Levi's approach, which is to use physics to find clever and succinct solutions. We claim this solution to be, at least, short.

6 Conclusion

Identifying the constraints of a constraint based geometry system with the load bearing constraints in a Lagrangian formulation of mechanics provides a system capable of succinct expressions of core engineering problems, such that symbolic results may be derived which are simple enough to be useful. Where sufficient simplicity cannot be achieved, symbolic results may be provided in the form of computer code for further analysis in numerical analysis environments and embedding in a program.

The system described in this paper is built on a geometry system which resolves a constraint model into a construction sequence typical of a dynamic geometry system. This places a restriction on the geometries which may be modeled in the system, but has the benefit of facilitating symbolic solution. The identification of geometric constraint with physical constraint implies that a reconfiguration of the geometric constraints to ensure constructibility may misalign them with intended structural elements of the model. For given geometry, applied forces and constraint reactions satisfy a linear system. Force elements of indeterminate strength may be added to correspond to physical constraints which are not geometrical. A linear system may be set up to find the appropriate values for these elements such that the geometric constraints which are not physical bear no load.

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